A NOTE ON BRUHAT ORDER AND DOUBLE COSET REPRESENTATIVES

CHRISTOPHE HOHLWEG AND MARK SKANDERA

1. Introduction

Let (W, S) be a finite Coxeter system with *length function* ℓ and identity e. Endow W with the *Bruhat order* $\leq$, that is, $w \leq g$ in W if and only if an expression for w can be obtained by deleting simple reflections in a reduced expression for g. (If $w \leq g$ then we necessarily have $\ell(w) \leq \ell(g)$.) We refer the reader to [2, 5] as general references for Coxeter groups and the Bruhat order.

Denote by W_I the *standard parabolic subgroup* of W generated by $I \subset S$. For $I, J \subset S$, each double coset in $W_I \backslash W / W_J$ has a unique minimal element. Let $X_{IJ} = \{w \in W \,|\, w < rw,\ w < ws,\ \forall r \in I, \forall s \in J\}$ be the set of all minimal representatives of double cosets in $W_I \backslash W / W_J$.

Curtis [3, Theorem 1.2] shows that for any $I, J \subset S$ and $b \in X_{IJ}$, there is a unique maximal element $b^{\max}$ in $W_I b W_J$. This fact plays an important role in his study of Lusztig's isomorphism theorem.

The aim of this note is to prove the following result, which seems to have escaped observation:

Theorem 1. *Let $I, J \subset S$ and $u, v \in X_{IJ}$. Then $u \leq v$ if and only if $u^{\max} \leq v^{\max}$.*

Double parabolic cosets arise in a variety of settings. In particular Theorem 1 is used in [7] in the study of the dual canonical basis of $\mathcal{O}(SL_n\mathbb{C})$.

After proving our result in §2, we give a combinatorial criterion in §3 for the comparison of u and v (or $u^{\max}$ and $v^{\max}$.)

2. Proof

For $I \subset S$, it is well-known that the set

$$W^I = \{u \in W \,|\, u < us, \forall s \in I\}$$

is a set of minimal length coset representatives of W/W_I. Each element $w \in W$ has therefore a unique decomposition $w = w^I w_I$ where $w^I \in W^I$ and $w_I \in W_I$. Moreover $\ell(w) = \ell(w^I) + \ell(w_I)$. The pair (w^I, w_I) is generally refered to as the *parabolic components of w along I* (see [2, Proposition 2.4.4], or [5, 5.12]). It is clear that $W_I^K = W^K \cap W_I$ is a set of minimal length coset representatives of W_I/W_K. Moreover $X_{IJ} = (W^I)^{-1} \cap W^J$, where $(W^I)^{-1} = \{w^{-1} \,|\, w \in W^I\}$.

Let $w_{0,I}$ denote the unique maximal element in in W_I, and let $w_0 = w_{0,S}$ denote the longest element of W. Then the parabolic components of w_0 are $(w_0^I, w_{0,I})$, where w_0^I is the unique maximal element in W^I (see [2, §2.5]). It follows that for $K \subset I \subset S$, the unique maximal element in W_I^K is $w_{0,I}^K = w_{0,I} w_{0,K}$.

We recall the following well-known facts:

(i) For any $I, J \subset S$, define $I \cap bJb^{-1} = I \cap \{bsb^{-1} \mid s \in J\}$. Then we have

$$W^J = \coprod_{b \in X_{IJ}} W_I^{I \cap bJb^{-1}} b.$$

Therefore, each element $w \in W$ has a unique decomposition (a, b, w_J) where $b \in X_{IJ}$, $a \in W_I^{I\cap bJb^{-1}}$ and $ab = w^J$. Moreover $\ell(w) = \ell(a) + \ell(b) + \ell(w_J)$. (See for instance [1, §2].)

(ii) Let $w, g, x \in W$ satisfy $\ell(wx) = \ell(w) + \ell(x)$ and $\ell(gx) = \ell(g) + \ell(x)$. Then $w \leq g$ if and only if $wx \leq gx$.

(iii) From (ii) we have: if $w \leq g$, then $w^I \leq g^I$ for any $w, g \in W$ and $I \subset S$.

(iv) *Deodhar's Lemma* [4]: Let $K \subset S$, $x \in W^K$ and $s \in S$. If $sx < x$ then $sx \in W^K$. If $x < sx$ then either $sx \in W^K$ or $sx = xr$ with $r \in K$.

(v) *Lifting property*: Let $w, g \in W$ and $s \in S$ satisfy $w < sw$ and $sg < g$. Then $w \leq g \iff w \leq sg \iff sw \leq g$ (see [5, 2]).

Curtis [3, Theorem 1.2] shows that for any $I, J \subset S$ and $b \in X_{IJ}$, $b^{\max} = w_{0,I}^{I\cap bJb^{-1}} b w_{0,J}$ is the unique maximal element in $W_I b W_J$. Here we give a short proof of this fact. Let $w \in W_I b W_J$, then by (i) we have $w = abw_J$ with $a \in W_I^{I\cap bJb^{-1}}$. Hence $a \leq w_{0,I}^{I\cap bJB^{-1}}$ and $w_J \leq w_{0,J}$, and by (i) and (ii) we have $w \leq b^{\max}$.

Lemma 2. *Let $I, J \subset S$ and suppose that $u, v \in X_{IJ}$ satisfy $u \leq v$. Then for any $a \in W_I^{I\cap uJu^{-1}}$ we have $au \leq w_{0,I}^{I\cap vJv^{-1}} v$.*

Proof. Writing $\alpha = w_{0,I}^{I\cap vJv^{-1}}$, we will use induction on $\ell(a)$ to show that $au \leq \alpha v$. If $\ell(a) = 0$, then $a = e$ and $au = u \leq \alpha v$ since $u \leq v$ and $\ell(\alpha v) = \ell(\alpha) + \ell(v)$.

Assume therefore that $\ell(a) > 0$. Then some $s \in I$ satisfies $sa < a$ and we have $sa \in W_I^{I\cap uJu^{-1}}$ by Deodhar's Lemma. Since $\ell(sa) < \ell(a)$, we have $sau \leq \alpha v$ by our induction hypothesis. We also have $sau < au$ by (ii) since $a, sa \in W_I$ and $u^{-1} \in W^I$. In order to compare αv and au we consider two cases.

If $s\alpha v < \alpha v$ then we obtain $au \leq \alpha v$ from (v) using $w = sau$ and $g = \alpha v$. If $\alpha v < s\alpha v$, then $au = s(sau) \leq s\alpha v$ by definition. Observe that $\alpha < s\alpha$ by (ii). As α is the maximal element in $W_I^{I\cap vJv^{-1}}$, we have $s\alpha \notin W_I^{I\cap vJv^{-1}}$ by (iv). So some $r \in I \cap vJv^{-1}$ satisfies $s\alpha = \alpha r$. Set $t = v^{-1}rv \in J$ so that $s\alpha v = \alpha v t$. As $\alpha v \in W^J$ (by (i)) we deduce that $(\alpha v, t)$ are the parabolic components of $s\alpha v$ along J. As $au \in W^J$ and $au \leq s\alpha v$, we obtain by (iii) that $au = (au)^J \leq (s\alpha v)^J = \alpha v$. $\square$

Proof of Theorem 1. Assume that $u^{\max} \leq v^{\max}$. First observe that from (1) and (i) we have $(((b^{\max})^J)^{-1})^I = b^{-1}$ for any $b \in X_{IJ}$. Now use (iii) and the automorphism $w \mapsto w^{-1}$ of the Bruhat order to show that $u^{\max} \leq v^{\max}$ implies $u \leq v$.

Assume now that $u \leq v$. Using (1), (i) and (ii) we just have to show that $w_{0,I}^{I\cap uJu^{-1}} u = (u^{\max})^J \leq (v^{\max})^J = w_{0,I}^{I\cap vJv^{-1}} v$. But this is the special case $a = w_{0,I}^{I\cap uJu^{-1}}$ of Lemma 2. $\square$

3. The special case of the symmetric group

Given a permutation $w \in S_n$, we define the matrix $M(w) = (m_{i,j}(w))$ by setting $m_{i,j}(w) = \delta_{j,w_i}$, where $w_1 \cdots w_n$ is the one-line notation of w. We define the related matrix $D(w) = (d_{i,j}(w))$ by $d_{i,j}(w) = \sum_{k=1}^{i} \sum_{\ell=1}^{j} m_{i,j}(w)$. It is well known that $u \leq v$ if and only if we have the componentwise inequality of matrices $D(u) \geq D(v)$, and we shall state a similar fact for double parabolic analogs of M and D.

A subset H of $S = \{s_1, \dots, s_{n-1}\}$ induces an equivalence relation $\sim_H$ on $[n] = \{1, \dots, n\}$ which is the transitive closure of the relation $i \, R \, (i{+}1)$ for all $s_i \in H$. Let $B_1, \dots, B_p$ and $C_1, \dots, C_q$ be the equivalence classes of $\sim_I$ and $\sim_J$, respectively, and define the matrices $M^{I,J}(w) = (m^{I,J}_{i,j}(w))$ and $D^{I,J}(w) = (d^{I,J}_{i,j}(w))$ by

$$m^{I,J}_{i,j}(w) = \#\{k \in B_i \,|\, w_k \in C_j\}, \qquad d^{I,J}_{i,j}(w) = \sum_{k=1}^{i} \sum_{\ell=1}^{j} m^{I,J}_{i,j}(w).$$

It is well known (see, e.g., [6]) that u and v belong to the same double coset in $W_I \backslash W / W_J$ if and only if $M^{I,J}(u) = M^{I,J}(v)$. Furthermore we have the following.

Proposition 3. *Given u, v in $X_{I,J}$, then $u \leq v$ (or $u^{\max} \leq v^{\max}$) if and only if we have the componentwise inequality of matrices $D^{I,J}(u) \geq D^{I,J}(v)$.*

Proof. Define $\overline{I} = [n] \smallsetminus \{i \,|\, s_i \in I\}$, $\overline{J} = [n] \smallsetminus \{j \,|\, s_j \in J\}$. Then for each $w \in S_n$, the matrix $D^{I,J}(w)$ is equal to the $(\overline{I}, \overline{J})$ submatrix of $D(w)$. The "only if" direction follows immediately.

Suppose the $u \not\leq v$ and let (i, j) be a componentwise minimal pair satisfying $d_{i,j}(u) < d_{i,j}(v)$. If $i > 1$, then the fact that the matrices $D(u)$ and $D(v)$ weakly increase down columns and across rows, with adjacent entries differing by no more than 1, implies that $d_{i-1,j}(u) \leq d_{i,j}(u) < d_{i,j}(v) \leq d_{i-1,j}(v){+}1$. By the minimality of i and j, this last expression is less than or equal to $d_{i-1,j}(u) + 1$, and for some nonnegative integer c we have $d_{i-1,j}(u) = d_{i,j}(u) = d_{i-1,j}(v) = c$ and $d_{i,j}(v) = c{+}1$. Similarly, if $j > 1$ then we have $d_{i,j-1}(u) = d_{i,j}(u) = d_{i,j-1}(v) = c$. It follows that for any values of (i, j) we must have $u_i > j$ and $u^{-1}_j > i$.

Now let (k, ℓ) be the componentwise minimal pair in $\overline{I} \times \overline{J}$ satisfying $i \leq k$, $j \leq \ell$. Since $u \in X_{I,J}$, we must also have

$$\ell < u_i < \cdots < u_k, \qquad k < u^{-1}_j < \cdots < u^{-1}_\ell.$$

Thus $d_{k,\ell}(u) = c$. Since $d_{k,\ell}(v) \geq c + 1$, we conclude that $D^{I,J}(u) \not\geq D^{I,J}(v)$. The equivalence of $D^{I,J}(u) \geq D^{I,J}(v)$ and $u^{\max} \leq v^{\max}$ follows from a similar argument. $\square$

We illustrate Proposition 3 by considering $W = S_7$, subsets $I = \{s_1, s_2, s_4, s_6\}$, $J = \{s_1, s_3, s_4, s_5\}$ of generators, and corresponding equivalence classes 123|45|67, 12|3456|7. To compare minimal representatives $u = 1342567$, $v = 3471526$ of two double cosets in $W_I \backslash W / W_I$, we use the matrices

$$M^{I,J}(u) = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad M^{I,J}(v) = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

to compute

$$D^{I,J}(u) = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 5 & 5 \\ 2 & 6 & 7 \end{bmatrix}, \qquad D^{I,J}(v) = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 4 & 5 \\ 2 & 6 & 7 \end{bmatrix},$$

and conlcude that $u \leq v$ and $u^{\max} \leq v^{\max}$.

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(Christophe Hohlweg) The Fields Institute, 222 College Street, Toronto, Ontario, M5T 3J1, CANADA

E-mail address: `chohlweg@fields.utoronto.ca`

URL: `http://www.fields.utoronto.ca/~chohlweg`

(Mark Skandera) Mathematics Department, Haverford College, 370 Lancaster Ave, Haverford, PA 19041, USA

E-mail address: `mskander@haverford.edu`

URL: `http://www.haverford.edu/math/skandera.html`